# Hausaufgabenblatt 5 (Lösungen)

**Aufgabe 5.1** (Integrale I)**.** Berechnen Sie.

**(a)** $\displaystyle\int_0^2 x^3 \, \mathrm{d}x$

***Lösung*:**

$$\int_0^2 x^3 \, \mathrm{d}x = \frac{x^4}{4}\bigg|_{x=0}^{2} = \frac{2^4 - 0^4}{4} = \underline{\underline{4}}.$$

**(b)** $\displaystyle\int_0^\infty e^{-x} \, \mathrm{d}x$

***Lösung*:**

$$\int_0^\infty e^{-x} \, \mathrm{d}x = \lim_{a\to\infty} \int_0^a e^{-x} \, \mathrm{d}x = \lim_{a\to\infty} -e^{-x}\Big|_{x=0}^{a} = \lim_{a\to\infty} -e^a + e^0 = \underline{\underline{1}}.$$

**(c)** $\displaystyle\int_{-1}^1 \sinh x \, \mathrm{d}x$

***Lösung*:** Mit $\sinh x = \frac{e^x - e^{-x}}{2}$ bekommen wir

$$\int_{-1}^1 \sinh x \, \mathrm{d}x = \int_{-1}^1 \frac{e^x - e^{-x}}{2} \, \mathrm{d}x = \frac{e^x + e^{-x}}{2}\bigg|_{x=-1}^{1} = \frac{e + e^{-1}}{2} - \frac{e^{-1} + e}{2} = \underline{\underline{0}}.$$

**(d)** $\displaystyle\int_0^{\pi/2} \cos x \, \mathrm{d}x$

***Lösung*:**

$$\int_0^{\pi/2} \cos x \, \mathrm{d}x = \sin x\Big|_{x=0}^{\pi/2} = \sin(\pi/2) - \sin 0 = \underline{\underline{1}}.$$

**(e)** $\displaystyle\int_{e^{-1}}^{e^2} \frac{1}{x} \, \mathrm{d}x$

***Lösung*:**

$$\int_{e^{-1}}^{e^2} \frac{1}{x} \, \mathrm{d}x = \ln x\Big|_{x=e^{-1}}^{e^2} = \ln e^2 - \ln e^{-1} = 2 + 1 = \underline{\underline{3}}.$$

**Aufgabe 5.2** (Integrale II)**.** Berechnen Sie durch partielle Integration.

**(a)** $\displaystyle\int_0^\pi x^2 \cdot \cos(x)\ \mathrm{d}x$

***Lösung:***

$$\begin{aligned}\int_0^\pi x^2 \cdot \cos(x)\ \mathrm{d}x &= x^2 \cdot \sin(x)\Big|_{x=0}^{\pi} - \int_0^\pi 2x \cdot \sin(x)\ \mathrm{d}x \\ &= 0 - \int_0^\pi 2x \cdot \sin(x)\ \mathrm{d}x \\ &= -2x(-\cos(x))\Big|_{x=0}^{\pi} + \int_0^\pi 2(-\cos(x))\ \mathrm{d}x \\ &= -2\sin(x)\Big|_{x=0}^{\pi} - 2\pi = \underline{\underline{-2\pi}}\end{aligned}$$

**(b)** $\displaystyle\int_{-1}^1 x^3 \cdot \exp(x)\ \mathrm{d}x$

***Lösung:***

$$\begin{aligned}\int_{-1}^1 x^3 \cdot \exp(x)\ \mathrm{d}x &= x^3\exp(x)\Big|_{x=-1}^{1} - \int_{-1}^1 3x^2\exp(x)\ \mathrm{d}x \\ &= x^3\exp(x)\Big|_{x=-1}^{1} - 3x^2\exp(x)\Big|_{x=-1}^{1} + \int_{-1}^1 6x\exp(x)\ \mathrm{d}x \\ &= x^3\exp(x)\Big|_{x=-1}^{1} - 3x^2\exp(x)\Big|_{x=-1}^{1} + 6x\exp(x)\Big|_{x=-1}^{1} - \int_{-1}^1 6\exp(x)\ \mathrm{d}x \\ &= (x^3 - 3x^2 + 6x - 6)\exp(x)\Big|_{x=-1}^{1} = \underline{\underline{16e^{-1} - 2e}}\end{aligned}$$

**(c)** $\displaystyle\int_0^\pi \sin(x)^2\ \mathrm{d}x$

***Lösung:***

$$\begin{aligned}\int_0^\pi \sin(x)^2\ \mathrm{d}x &= \int_0^\pi \sin(x)\cdot\sin(x)\ \mathrm{d}x \\ &= -\cos(x)\sin(x)\Big|_{x=0}^{\pi} - \int_0^\pi (-\cos(x))\cdot\cos(x)\ \mathrm{d}x \\ &= \int_0^\pi \cos(x)^2\ \mathrm{d}x = \int_0^\pi 1 - \sin(x)^2\ \mathrm{d}x \qquad \Big| + \int_0^\pi \sin(x)^2\ \mathrm{d}x\end{aligned}$$

$$2\int_0^\pi \sin(x)^2 \, \mathrm{d}x = \int_0^\pi 1 \, \mathrm{d}x = \pi \qquad \Big| \div 2$$

$$\int_0^\pi \sin(x)^2 \, \mathrm{d}x = \underline{\underline{\frac{\pi}{2}}}$$

**Aufgabe 5.3** (Integrale III)**.** Berechnen Sie durch Substitution.

**(a)** $\displaystyle\int_0^\pi x^2 \cdot \cos(x^3) \, \mathrm{d}x$

***Lösung:*** Setze $y = x^3$, d.h. $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$ also $x^2 \, \mathrm{d}x = \frac{1}{3}\mathrm{d}y$. Integrationsgrenzen: $x_1 = 0 \mapsto y_1 = x_1^3 = 0^3 = 0$ und $x_2 = \pi \mapsto y_2 = x_2^3 = \pi^3$. Es folgt

$$\int_0^\pi x^2 \cdot \cos(x^3) \, \mathrm{d}x = \int_0^{\pi^3} \cos(y)\frac{1}{3}\mathrm{d}y = \frac{1}{3}\sin(y)\Big|_{y=0}^{\pi^3} = \frac{1}{3}(\sin(\pi^3) - \sin(0)) = \underline{\underline{\frac{\sin(\pi^3)}{3}}}$$

**(b)** $\displaystyle\int_1^e \ln\left((x^2+1)^x\right) \, \mathrm{d}x.$

Hinweis: Setze $y = x^2 + 1$.

***Lösung:*** Setze $y = x^2 + 1$, d.h. $x \, \mathrm{d}x = \frac{1}{2}\mathrm{d}y$ mit den Grenzen $y_1 = x_1^2 + 1 = 2$ und $y_2 = x_2^2 + 1 = e^2 + 1$. Also

$$\begin{aligned}\int_1^e \ln\left((x^2+1)^x\right) \, \mathrm{d}x &= \int_1^e x\ln\left(x^2+1\right) \, \mathrm{d}x = \frac{1}{2}\int_2^{e^2+1} \ln(y) \, \mathrm{d}y \\ &= y\ln(y) - y\Big|_{y=2}^{e^2+1} = (e^2+1)\ln(e^2+1) - (e^2+1) - 2\ln 2 + 2 \\ &= \underline{\underline{(e^2+1)\ln(e^2+1) - 2\ln 2 - e^2 + 1}}.\end{aligned}$$

**(c)** $\displaystyle\int_{-1}^1 \frac{1}{1+e^x} \, \mathrm{d}x.$

Hinweis: Erweitere den Bruch mit $e^{-x}$ und substituiere $y = e^{-x} + 1$.

***Lösung:*** Setze $y = e^{-x} + 1$, d.h. $e^{-x} \, \mathrm{d}x = -\mathrm{d}y$ und $y_1 = e + 1$ und $y_2 = e^{-1} + 1$ somit

$$\int_{-1}^1 \frac{1}{1+e^x} \, \mathrm{d}x = \int_{-1}^1 \frac{e^{-x}}{e^{-x}+1} \, \mathrm{d}x = -\int_{e+1}^{e^{-1}+1} \frac{1}{y} \, \mathrm{d}y$$

$$= -\ln(y)\Big|_{y=e+1}^{e^{-1}+1}$$
$$= \underline{\underline{\ln\left(\frac{e+1}{e^{-1}+1}\right)}}.$$